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# On the structure of the observable algebra of QCD on the lattice

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## Abstract

The structure of the observable algebra  $\mathfrak{D}_\Lambda$  of lattice QCD in the Hamiltonian approach is investigated. As was shown earlier,  $\mathfrak{D}_\Lambda$  is isomorphic to the tensor product of a gluonic  $C^*$ -subalgebra, built from gauge fields and a hadronic subalgebra constructed from gauge-invariant combinations of quark fields. The gluonic component is isomorphic to a standard CCR algebra over the group manifold  $SU(3)$ . The structure of the hadronic part, as presented in terms of a number of generators and relations, is studied in detail. It is shown that its irreducible representations are classified by triality. Using this, it is proved that the hadronic algebra is isomorphic to the commutant of the triality operator in the enveloping algebra of the Lie superalgebra  $sl(1/n)$  (factorized by a certain ideal).

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## 1. Introduction

During the last decades, quantum chromodynamics (QCD) has become one of the basic building blocks of the standard model for describing elementary particle interactions. It was quite successful, e.g. in describing deep inelastic scattering processes within perturbation theory on the one hand and ‘measuring’ certain types of observables using (nonperturbative) lattice approximation techniques on the other hand; see [1–3] for pioneering work in this direction. We stress that also within the programme of constructive quantum gauge field theory, lattice methods are of fundamental importance; see [4] and references therein.

Despite all efforts made, we are still facing a basic challenge, which consists of constructing an effective microscopic theory of interacting hadrons out of QCD. This is certainly a nonperturbative problem. To solve it, in our opinion, one should start with a

careful analysis of the concept of observables. The lattice approximation seems to be an ideal intermediate step for that purpose. (Finally, of course, one would like to construct the continuum limit—an extraordinarily complicated problem of constructive quantum field theory.) To be rigorous, we have put QCD on a *finite* (regular cubic) lattice and we have started investigating it in the Hamiltonian approach, see [5, 6].

In [5] we have analysed the structure of the field algebra of QCD and the Gauss law. Comparing with quantum electrodynamics (QED), in QCD the local Gauss law is neither built from gauge-invariant operators nor is it linear. But it is possible to extract a gauge invariant, additive law for operators with eigenvalues in the dual of the centre of  $SU(3)$ , (which is identified with  $\mathbb{Z}_3$ ). This implies—as in QED—a gauge-invariant conservation law: the global  $\mathbb{Z}_3$ -valued colour charge (trinality) is equal to a  $\mathbb{Z}_3$ -valued gauge-invariant quantity obtained from the colour electric flux at infinity. We stress that the notion of triality occurred in the literature a long time ago. On the level of lattice gauge theories, this notion is already implicitly contained in a paper by Kogut and Susskind, see [2]. In particular, Mack [7] used it to propose a certain (heuristic) scheme of colour screening and quark confinement, based upon a dynamical Higgs mechanism with Higgs fields built from gluons. For similar ideas we also refer to papers by 't Hooft, see [8] and references therein, and Cornwall, see [9]. There is a whole series of papers by the latter author, where the centre vortex idea has been further developed, see [10] and references therein. The triality concept was also used in a paper by Borgs and Seiler [11] on the confinement problem for Yang–Mills theories with static quark sources at nonzero temperature. In this context, also the Gauss law for colour charge was analysed.

In [6] we analysed the observable algebra of QCD in the above context. The  $C^*$ -algebra  $\mathfrak{D}_\Lambda^i$  of physical observables, internal relative to  $\Lambda$  is, by definition, the algebra of gauge-invariant fields, with the Gauss law imposed. In order to take into account correlations with the 'rest of the world', one has to supplement  $\mathfrak{D}_\Lambda^i$  by the algebra  $\mathfrak{D}_\Lambda^\infty$  of gauge-invariant elements at infinity. The full algebra of observables  $\mathfrak{D}_\Lambda$  is the tensor product of these two pieces. In [6] we have shown that there are three inequivalent representations of  $\mathfrak{D}_\Lambda$  labelled by global colour charge. It turns out that  $\mathfrak{D}_\Lambda$  is isomorphic to the tensor product of a gluonic  $C^*$ -subalgebra, built from lattice gauge fields and a hadronic subalgebra constructed from gauge-invariant combinations of quark fields. The gluonic component is isomorphic to a standard CCR algebra over the group manifold  $SU(3)$ , whereas the hadronic subalgebra  $\mathfrak{D}_T^{\text{mat}}$  is built from bilinear and trilinear gauge-invariant combinations of the quark fields.

The present paper is a continuation of [5] and [6]. It is motivated by the following points.

- (i) For purposes of quantum theory, one needs a complete presentation of  $\mathfrak{D}_\Lambda$  in terms of generators and relations. Thus, we present a complete analysis of the structure of the hadronic subalgebra, selecting, in particular, a minimal set of relations.
- (ii) The classification of irreducible representations of  $\mathfrak{D}_\Lambda$  presented in [6] was in abstract terms. It was a challenge, to prove the same result in the language of generators and relations. Here, we show that this can be done indeed, the key observation being that the hadronic subalgebra is isomorphic to the commutant of the triality operator in the CAR algebra generated by  $n = 12N$  creation and annihilation operators ( $N$  is the number of lattice sites).
- (iii) A similar analysis for (spinorial and scalar) QED has been presented in [14–16]. There, the matter field part of the observable algebra is generated by a certain Lie algebra, a fact which should be helpful for constructing the thermodynamical limit in the future. Does a similar structure occur in QCD too? Here, we prove that the hadronic subalgebra is generated, in a certain sense, by a standard Lie superalgebra. Moreover, in this

language, the three inequivalent representations naturally arise via a standard Kac module construction.

The presentation of  $\mathfrak{D}_\Lambda$  used in this paper is based upon a certain gauge fixing procedure, which works well on the generic stratum of the action of the gauge group on the underlying classical configuration space. First steps towards including non-generic strata have been made as well [12]. It is worthwhile to try to omit the gauge fixing philosophy and to analyse  $\mathfrak{D}_\Lambda$  in more intrinsic terms. This leads to polynomial superalgebras, see [13].

Finally, we note that standard methods from algebraic quantum field theory for models which do not contain massless particles, see [17], do not apply here. For an analysis of problems with massless particles within this approach we refer to [18–23] and further references therein.

Our paper is organized as follows: In section 2, we briefly summarize the results of [5, 6]. In subsection 3.1, we present a systematic study of the hadronic subalgebra  $\mathfrak{D}_T^{\text{mat}}$  in terms of generators and relations. Next, in subsection 3.2, we reduce the set of relations to a certain minimal set and in subsection 3.3 we present a classification of irreducible representations of  $\mathfrak{D}_T^{\text{mat}}$ . Finally, in subsection 3.4, the above-mentioned super Lie structure is presented.

## 2. The full observable algebra

Here, we briefly summarize the results of [5, 6].

We consider QCD in the Hamiltonian framework on a finite regular cubic lattice  $\Lambda \subset \mathbb{Z}^3$ , with  $\mathbb{Z}^3$  being the infinite regular lattice in three dimensions. We denote the lattice boundary by  $\partial\Lambda$  and the set of oriented,  $j$ -dimensional elements of  $\Lambda$ , respectively  $\partial\Lambda$ , by  $\Lambda^j$ , respectively  $\partial\Lambda^j$ , where  $j = 0, 1, 2, 3$ . Such elements are (in increasing order of  $j$ ) called sites, links, plaquettes and cubes. Moreover, we denote the set of external links connecting boundary sites of  $\Lambda$  with ‘the rest of the world’ by  $\Lambda_\infty^1$  and the set of endpoints of external links at infinity by  $\Lambda_\infty^0$ . Assume that for each boundary site there is exactly one link with infinity. Then, external links are labelled by boundary sites and we can denote them by  $(x, \infty)$  with  $x \in \partial\Lambda^0$ .

The basic fields of lattice QCD are quarks  $\psi^{aA}$ , together with their conjugates  $\psi_{aA^*}$ , living at lattice sites, and gluonic gauge potentials  $U^A_B$ , together with their conjugate momenta  $E^A_B$  (colour electric fields), living on links (including links connecting the lattice under consideration with ‘infinity’). Here,  $a$  stands for bispinorial and (possibly) flavour degrees of freedom and  $A, B, \dots = 1, 2, 3$  denote the colour index. For these generators, the canonical (anti)-commutation relations are postulated. Moreover, the generators fulfil further relations according to their algebraic nature. In [5, 6], we have analysed the structure of the *field algebra*  $\mathfrak{A}_\Lambda$ , defined by these generators and relations, in detail. Moreover, we have shown that  $\mathfrak{A}_\Lambda$  has a unique (up to unitary equivalence) irreducible representation (generalized Schrödinger representation). Via this representation, the field algebra  $\mathfrak{A}_\Lambda$  can be identified with the algebra  $\mathfrak{K}(H_\Lambda)$  of compact operators on the Hilbert space

$$H_\Lambda = \mathcal{F}(\mathbb{C}^{12N}) \otimes L^2(\mathcal{C}, \mu). \tag{2.1}$$

Here,  $\mathcal{F}(\mathbb{C}^{12N})$  denotes the fermionic Fock space generated by  $12N$  anti-commuting pairs of quark fields and

$$\mathcal{C} := \prod_{(x,y) \in \Lambda^1} \mathcal{C}_{(x,y)} \prod_{x \in \partial\Lambda^0} \mathcal{C}_{(x,\infty)},$$

with  $\mathcal{C}_{(x,y)}$  being diffeomorphic to the group space  $G$  ( $\mu$  is the product of Haar measures).

The group of local gauge transformations  $G_\Lambda$  related to the lattice  $\Lambda$  consists of mappings  $\Lambda^0 \ni x \rightarrow g(x) \in G$ , which represent internal gauge transformations, and of gauge transformations at infinity,  $\Lambda_\infty^0 \ni z \rightarrow g(z) \in G$ . Thus,

$$G_\Lambda := G_i \times G_\infty = \prod_{x \in \Lambda^0} G_x \prod_{z \in \Lambda_\infty^0} G_z, \quad (2.2)$$

with  $G_y \cong SU(3)$ , for every  $y$ . This group acts by automorphisms on the field algebra  $\mathfrak{A}_\Lambda$ . Imposing gauge invariance on physical states yields the following *local Gauss law* at  $x \in \Lambda^0$ :

$$\sum_{y \leftrightarrow x} E^A_B(x, y) = \rho^A_B(x), \quad (2.3)$$

where  $\rho^A_B(x)$  is the local matter charge at  $x$ . In [5] we have shown that, for every integrable representation  $F$  of  $su(3)$ , there exists a  $\mathbb{Z}_3$ -valued operator function  $F \rightarrow \varphi(F)$ , which is additive:

$$\varphi(F + G) = \varphi(F) + \varphi(G), \quad (2.4)$$

for all  $F$  and  $G$  commuting. Here,  $\mathbb{Z}_3 \cong \{-1, 0, 1\}$  stands for the group of characters on the centre  $\mathcal{Z}$  of  $SU(3)$ . Of course, any  $F$  can be decomposed into its irreducible components. If  $F$  is irreducible, characterized by highest weight  $(m, n)$ , one gets

$$\varphi(F) = (m - n) \pmod{3}. \quad (2.5)$$

Applying  $\varphi$  to the local Gauss law (2.3) at every lattice point and taking the sum over all lattice sites, we get the *global Gauss law*

$$\Phi_{\partial\Lambda} = \mathfrak{t}_\Lambda. \quad (2.6)$$

Here,  $\Phi_{\partial\Lambda} = \sum_{x \in \partial\Lambda} \varphi(E(x, \infty))$  is the total flux through the boundary  $\partial\Lambda$  and  $\mathfrak{t}_\Lambda = \sum_{x \in \Lambda} \varphi(\rho(x))$  is the (gauge-invariant) global colour charge (*triality*), carried by the matter field.

The  $C^*$ -algebra  $\mathfrak{D}_\Lambda^i$  of physical observables, internal relative to  $\Lambda$  is, by definition, the algebra of gauge-invariant fields, with the Gauss law imposed. This means that we take the subalgebra of  $G_\Lambda$ -invariant elements of  $\mathfrak{A}_\Lambda$  and factorize it with respect to the ideal generated by local Gauss laws at all lattice sites. In order to take into account correlations with the ‘rest of the world’, one has to supplement  $\mathfrak{D}_\Lambda^i$  by the algebra  $\mathfrak{D}_\Lambda^\infty$  of gauge-invariant elements at infinity, yielding the *full algebra of observables*:

$$\mathfrak{D}_\Lambda \cong \mathfrak{D}_\Lambda^i \otimes \mathfrak{D}_\Lambda^\infty. \quad (2.7)$$

In [6] we have analysed the structure of  $\mathfrak{D}_\Lambda$  in detail. The main result of this paper was the proof that there are three inequivalent representations of  $\mathfrak{D}_\Lambda$  labelled by values of the global flux  $\Phi_{\partial\Lambda}$ . By the global Gauss law (2.6), these inequivalent representations can be labelled by eigenvalues of global colour charge  $\mathfrak{t}_\Lambda$ .

For purposes of quantum theory one needs, of course, an explicit presentation of  $\mathfrak{D}_\Lambda$  in terms of generators and relations. The algebra  $\mathfrak{D}_\Lambda^\infty$  is generated by operations of tensorizing or contracting with  $SU(3)$ -invariant tensors  $\delta^A_B, \epsilon^{ABC}$  and  $\epsilon_{ABC}$ , and by operators  $P^{(m,n)}$  projecting onto irreducible subspaces of the physical Hilbert space at infinity of valence  $(m, n)$ , see [6]. Thus, one has to find a complete set of generators of  $\mathfrak{D}_\Lambda^i$ . In [6] we have shown the following:

**Theorem 2.1.** *The observable algebra  $\mathfrak{D}_\Lambda^i$  is generated by the following gauge-invariant elements (together with their conjugates):*

$$U_\gamma := U_{\gamma A}^A \quad (2.8)$$

$$E_\gamma(x, y) := U_{\gamma B}^A E^B_A(x, y) \tag{2.9}$$

$$J_\gamma^{ab}(x, y) := \psi^{*a}_A(x) U_{\gamma B}^A \psi^{bB}(y) \tag{2.10}$$

$$W_{\alpha\beta\gamma}^{abc}(x, y, z) := \frac{1}{6} \epsilon_{ABC} U_{\alpha D}^A U_{\beta E}^B U_{\gamma F}^C \psi^{aD}(x) \psi^{bE}(y) \psi^{cF}(z), \tag{2.11}$$

with  $\gamma$  denoting an arbitrary closed lattice path in formula (2.8), a closed lattice path starting and ending at  $x$  in (2.9) and a path from  $x$  to  $y$  in (2.10). In formula (2.11),  $\alpha$ ,  $\beta$  and  $\gamma$  are paths starting at some reference point  $t$  and ending at  $x$ ,  $y$  and  $z$ , respectively. In formula (2.9), both  $x$  and  $y$  stand also for  $\infty$ .

Note that the observables  $J_\gamma^{ab}$  and  $W_{\alpha\beta\gamma}^{abc}$  represent hadronic matter of mesonic and baryonic type.

The above set of generators is, however, highly redundant. In a first step, it can be reduced by using the concept of a lattice tree. As a result, one obtains a presentation of the observable algebra in terms of tree data, which are still subject to gauge transformations at the (arbitrarily chosen) tree root. Finally, this gauge freedom has to be removed. This reduction procedure has been discussed in detail in [6]. The second step leads to delicate problems (Gribov problem and the occurrence of nongeneric strata), which suggest that one should investigate the stratified structure of the underlying classical configuration (resp. phase) space in more detail. See [12] for first results.

As a result of this reduction, the observable algebra  $\mathfrak{D}_\Lambda$  is obtained as

$$\mathfrak{D}_\Lambda = \mathfrak{D}_T^{\text{glu}} \otimes \mathfrak{D}_T^{\text{mat}} \otimes \mathfrak{D}_\Lambda^b \otimes \mathfrak{D}_\Lambda^\infty. \tag{2.12}$$

Here, the *gluonic* component  $\mathfrak{D}_T^{\text{glu}}$  is generated by reduced gluonic tree data  $(u_i, \epsilon_i)$ ,  $i = 0, \dots, K - 2$ , with  $K$  denoting the number of off-tree lattice links. These bosonic generators satisfy the generalized canonical commutation relations over  $G$ :

$$[\epsilon_{i s}^r, \epsilon_{j q}^p] = \delta_{ij} (\delta^p_s \epsilon_{i q}^r - \delta^r_q \epsilon_{i s}^p), \tag{2.13}$$

$$[\epsilon_{i s}^r, u_{j q}^p] = \delta_{ij} (\delta^p_s u_{i q}^r - \frac{1}{3} \delta^r_s u_{i q}^p), \tag{2.14}$$

$$[u_{i s}^r, u_{j q}^p] = 0, \tag{2.15}$$

with  $r, s, \dots = 1, 2, 3$ . The generators  $(u_i, \epsilon_i)$  are subject to a certain discrete symmetry described in [6] (which, however, is not a remainder of the gauge symmetry).

Applying the above gauge fixing procedure to the fermionic matter field, we obtain fermionic operators  $\phi_k$ . Here,  $k = (a, r, x)$  is a multi-index running from 1 to  $12N$ , with  $N$  being the number of lattice sites. These quantities fulfil the canonical anti-commutation relations

$$[\phi^{*k}, \phi_l]_+ = \delta^k_l. \tag{2.16}$$

(We stress that in [6], the generators  $\phi_k$  were denoted by  $\mathfrak{a}_k$ .) Again, an additional discrete symmetry arises, because the gauge fixing is defined only up to the stabilizer  $\mathbb{Z}_3$  of the generic stratum of the underlying classical configuration space. Thus, strictly speaking, the generators  $\phi_k$  are *not observables*, whereas the bosonic quantities  $u$  and  $\epsilon$  are, because they are not affected by this ambiguity. It is clear from classical invariants theory that only the following combinations of  $\phi^{*k}$  and  $\phi_k$  (together with functions built from them) are observables:

$$j_l^k = \phi^{*k} \phi_l, \tag{2.17}$$

$$i_l^k = \phi_l \phi^{*k}, \tag{2.18}$$

$$\mathfrak{w}_{pqr} = \phi_p \phi_q \phi_r, \tag{2.19}$$

$$\mathfrak{w}^{*ijk} = \phi^{*k} \phi^{*j} \phi^{*i}. \tag{2.20}$$

We have, of course,

$$i^k_l = \delta^k_l \mathbf{1} - j^k_l. \tag{2.21}$$

Thus, the matter field component  $\mathfrak{D}_T^{\text{mat}}$  is generated by the set  $\{j, \mathfrak{w}, \mathfrak{w}^*\}$ , together with the unit element  $\mathbf{1}$ . These generators are observables of hadronic type. Thus, in what follows we call  $\mathfrak{D}_T^{\text{mat}}$  *hadronic* component of the observable algebra or simply *hadronic subalgebra*.

Finally,  $\mathfrak{D}_\Lambda^b$  is generated by (gauge-invariant) colour electric boundary fluxes and the generators of  $\mathfrak{D}_\Lambda^\infty$  have been already given above.

By the uniqueness theorem for generalized CCR, fulfilled by generators  $(u_i, \epsilon_i)$ , of the gluonic subalgebra  $\mathfrak{D}_T^{\text{glu}}$ , the problem of classifying irreducible representations of  $\mathfrak{D}_\Lambda$  is reduced to classifying irreducible representations of the hadronic subalgebra  $\mathfrak{D}_T^{\text{mat}}$ . For that purpose, the structure of this algebra will be investigated in the following.

### 3. Structure of the hadronic subalgebra

#### 3.1. Generators and relations

We start analysing  $\mathfrak{D}_T^{\text{mat}}$  by listing relations implied from definitions (2.17)–(2.20).

First, note that  $\mathfrak{D}_T^{\text{mat}}$  is a unital  $*$ -algebra, with unit element  $\mathbf{1}$  and  $*$ -operation given by

$$(j^k_l)^* = j^l_k, \tag{3.1}$$

$$(\mathfrak{w}_{pqr})^* = \mathfrak{w}^{*rqp}, \tag{3.2}$$

$$(\mathfrak{w}^{*ijk})^* = \mathfrak{w}_{ijk}, \tag{3.3}$$

where the generators  $\mathfrak{w}$  and  $\mathfrak{w}^*$  are totally antisymmetric in their indices:

$$\mathfrak{w}_{mkn} = \mathfrak{w}_{knm} = \mathfrak{w}_{nmk} = -\mathfrak{w}_{kmn}. \tag{3.4}$$

Next, the anticommutation relations (2.16) immediately yield

$$[j^k_l, j^m_n] = \delta^m_l j^k_n - \delta^k_n j^m_l, \tag{3.5}$$

$$[j^i_k, \mathfrak{w}_{lmn}] = -\delta^i_l \mathfrak{w}_{kmn} - \delta^i_m \mathfrak{w}_{lkn} - \delta^i_n \mathfrak{w}_{lmk}, \tag{3.6}$$

and, consequently,

$$[j^k_i, \mathfrak{w}^{*lmn}] = \delta^l_i \mathfrak{w}^{*kmn} + \delta^m_i \mathfrak{w}^{*lkn} + \delta^n_i \mathfrak{w}^{*lmk}. \tag{3.7}$$

Observe that (3.5) are the commutation relations of  $\mathfrak{gl}(n, \mathbb{C})$ , with  $n = 12N$ . As a direct consequence of (2.17), these generators fulfil a number of additional quadratic relations: First, the diagonal generators  $j^k_k$  are idempotent

$$(j^k_k)^2 = j^k_k. \tag{3.8}$$

Because of the Hermiticity condition (3.1) they are, thus, projectors. Commutation relations (3.5) implies that they all commute with each other.

Finally, products of  $\mathfrak{w}$  and  $\mathfrak{w}^*$  can be expressed in terms of  $j$ 's:

$$\mathfrak{w}^{*kmn} \mathfrak{w}_{kmn} = j^k_k j^m_m j^n_n, \quad \text{for different } k, m, n, \tag{3.9}$$

$$\mathfrak{w}_{kmn} \mathfrak{w}^{*kmn} = i_k^k i_m^m i_n^n, \quad \text{for different } k, m, n. \tag{3.10}$$

We recall that

$$i_k^k = \mathbf{1} - j_k^k, \tag{3.11}$$

see (2.21). In the next subsection, we are going to prove that the above properties uniquely characterize the algebra  $\mathfrak{D}_T^{\text{mat}}$ .

We show that the triality operator belongs to the centre of the algebra. For this purpose, note that  $j_k^k$  is the particle number operator at position  $k$ . Thus,

$$\mathfrak{n} = \sum_{k=1}^n j_k^k \tag{3.12}$$

is the *total particle number operator*. By definition of  $\mathfrak{t}_\Lambda$ , we have

$$\mathfrak{t}_\Lambda = \varphi(\mathfrak{n}). \tag{3.13}$$

This means that, in any representation,  $\mathfrak{t}_\Lambda$  is equal to the particle number, modulo 3. As a direct consequence of (3.5), (3.6) and (3.7), we obtain

$$[\mathfrak{n}, j_l^k] = 0 \tag{3.14}$$

$$[\mathfrak{n}, \mathfrak{w}^{*ijk}] = 3\mathfrak{w}^{*ijk}, \tag{3.15}$$

$$[\mathfrak{n}, \mathfrak{w}_{pqr}] = -3\mathfrak{w}_{pqr}. \tag{3.16}$$

Together with (3.13), these relations imply that all generators and, thus, all hadronic observables commute with the triality operator  $\mathfrak{t}_\Lambda$ .

**Remark.** Due to (3.6), the whole set of baryonic invariants  $\mathfrak{w}_{lmn}$  can be generated from one chosen  $\mathfrak{w}_{l_0m_0n_0}$  by successively taking commutators with  $j$ 's. Indeed, for  $i \neq l$ , commutation relation (3.6) reduces to the identity  $[j_k^l, \mathfrak{w}_{lmn}] = -\mathfrak{w}_{kmn}$  which enables us to ‘flip’ the multi-index  $(l, m, n)$  to any other position. All relations for the remaining  $\mathfrak{w}$ 's then follow from the relations for the single selected element  $\mathfrak{w}_{l_0m_0n_0}$ .

### 3.2. Axiomatic description

Consider now the abstract, unital  $*$ -algebra  $\mathcal{A}$ , generated by abstract elements  $j$ ,  $\mathfrak{w}$  and  $\mathfrak{w}^*$ , which fulfil relations (3.1)–(3.10). In the following, we shall prove that these relations define the algebra uniquely, i.e.  $\mathcal{A}$  is identical with the previously defined algebra  $\mathfrak{D}_T^{\text{mat}}$ . For this purpose, we derive from the defining relations of  $\mathcal{A}$  a number of additional identities.

**Theorem 3.1.** *The defining relations (3.1)–(3.10) of  $\mathcal{A}$  imply the following additional identities:*

1.

$$j_l^m j_l^k = \delta_l^k j_l^m, \tag{3.17}$$

$$j_l^k j_n^k = \delta_l^n j_n^k. \tag{3.18}$$

Thus, in particular, the off-diagonal generators are nilpotent:

$$(j_l^k)^2 = 0 \quad \text{for } k \neq l. \tag{3.19}$$



2.

$$j^i_k \mathfrak{w}_{lmn} = -j^i_l \mathfrak{w}_{kmn} = -j^i_m \mathfrak{w}_{lkn} = -j^i_n \mathfrak{w}_{lmk}. \quad (3.20)$$

$$\mathfrak{w}^{*lmn} j^k_i = -\mathfrak{w}^{*kmn} j^l_i = -\mathfrak{w}^{*lkn} j^m_i = -\mathfrak{w}^{*lmk} j^n_i. \quad (3.21)$$

In particular,  $j^i_k$  multiplied with  $\mathfrak{w}$  vanishes, if at least one of the indices of  $\mathfrak{w}$  coincides with  $k$ , e.g.

$$j^k_l \mathfrak{w}_{lmn} = 0, \quad \mathfrak{w}^{*lmn} j^l_k = 0. \quad (3.22)$$

3. The following identities hold:

$$j^l_l \mathfrak{w}_{lmn} = 0 = \mathfrak{w}_{lmn} i^l_l \quad (3.23)$$

$$i^l_l \mathfrak{w}_{lmn} = \mathfrak{w}_{lmn} = \mathfrak{w}_{lmn} j^l_l \quad (3.24)$$

$$\mathfrak{w}^{*lmn} j^l_l = 0 = i^l_l \mathfrak{w}^{*lmn} \quad (3.25)$$

$$\mathfrak{w}^{*lmn} i^l_l = \mathfrak{w}^{*lmn} = i^l_l \mathfrak{w}^{*lmn}. \quad (3.26)$$

4. The generators  $\mathfrak{w}$  and  $\mathfrak{w}^*$  are nilpotent:

$$(\mathfrak{w}^{*ijk})^2 = 0, \quad (3.27)$$

$$(\mathfrak{w}_{pqr})^2 = 0. \quad (3.28)$$

**Proof.**

1. To show relations (3.17), we have to prove

$$j^m_l j^k_l = 0, \quad \text{for } k \neq l, \quad (3.29)$$

$$j^m_l j^l_l = j^m_l. \quad (3.30)$$

From (3.5) we get

$$j^k_l = [j^k_l, j^l_l] = j^k_l j^l_l - j^l_l j^k_l. \quad (3.31)$$

Multiplying this relation by  $j^l_l$  to the left and using (3.8) yields

$$j^l_l j^k_l = j^l_l j^k_l j^l_l - j^l_l j^k_l,$$

or, inserting expression (3.31) for  $j^k_l$ ,

$$2j^l_l j^k_l = j^l_l j^k_l j^l_l = j^l_l (j^k_l j^l_l - j^l_l j^k_l) j^l_l = 0.$$

This proves (3.29) for the case  $m = l$ . Consequently, (3.31) reduces to

$$j^k_l = j^k_l j^l_l. \quad (3.32)$$

This proves relation (3.30). Finally, multiplying (3.30) by  $j^k_l$  to the left and using  $j^l_l j^k_l = 0$  yields (3.29) for  $m \neq l$ . The proof of relations (3.18) is completely analogous and, therefore, we omit it here.

2. We show the first relation in (3.20),

$$j^i_k \mathfrak{w}_{lmn} = -j^i_l \mathfrak{w}_{kmn}.$$

First, the commutation relations (3.6) imply

$$j^l_l \mathfrak{w}_{lmn} - \mathfrak{w}_{lmn} j^l_l = -\mathfrak{w}_{lmn}. \quad (3.33)$$

Multiplying this equation from both sides by  $j^l_l$  we obtain

$$j^l_l \mathfrak{w}_{lmn} j^l_l = 0.$$

Hence, multiplying (3.33) from the left by  $j^l_l$  yields

$$j^l_l \mathfrak{w}_{lmn} = -j^l_l \mathfrak{w}_{lmn},$$

or  $j^l_l \mathfrak{w}_{lmn} = 0$ . Multiplying this relation to the left by  $j^k_l$  and using (3.30) yields

$$j^k_l \mathfrak{w}_{lmn} = 0, \tag{3.34}$$

showing the special case (3.22). Now, multiplying the commutation relations (3.6) to the left by  $j^i_l$  and using (3.18) together with (3.34) gives  $\delta^i_l j^i_k \mathfrak{w}_{lmn} = -\delta^i_l j^i_l \mathfrak{w}_{kmn}$  or, equivalently,

$$j^i_k \mathfrak{w}_{lmn} = -j^i_l \mathfrak{w}_{kmn}. \tag{3.35}$$

Finally, multiplying this equation by  $j^i_l$  and using (3.6), (3.34) and (3.30) yields the proof of the statement. The proof of the remaining equations contained in (3.20) is identical.

3. First, observe that by (3.34), the auxiliary identity (3.33) reduces to

$$-\mathfrak{w}_{lmn} j^l_l = -\mathfrak{w}_{lmn}, \tag{3.36}$$

and, hence, we have  $\mathfrak{w}_{lmn} i^l_l = 0$ . This way (3.23) and (3.24) are proved. Acting with the operator  $*$  on both sides we obtain the remaining identities (3.25) and (3.26).

4. Identity (3.23) and (3.24) imply nilpotency of  $\mathfrak{w}$ :

$$\mathfrak{w}_{lmn} \mathfrak{w}_{lmn} = \mathfrak{w}_{lmn} (i^l_l \mathfrak{w}_{lmn}) = (\mathfrak{w}_{lmn} i^l_l) \mathfrak{w}_{lmn} = 0.$$

Similarly, nilpotency of  $\mathfrak{w}^*$  follows from the remaining two identities. □

Finally, observe that the idempotency and nilpotency properties (3.8) and (3.19) render  $\mathcal{A}$  finite dimensional. To summarize,  $\mathcal{A}$  is a (finite-dimensional) associative unital  $*$ -algebra. It is obtained from the free algebra, generated by elements  $\{j, \mathfrak{w}, \mathfrak{w}^*, \mathbf{1}\}$ , by factorizing with respect to the relations listed above.

For the sake of completeness, we have listed additional interesting identities, see the appendix, which have to be taken into account, if one wants to build arbitrary monomials in the generators.

### 3.3. Irreducible representations

**Lemma 3.2.** *There is at least one nontrivial, faithful irreducible representation of  $\mathcal{A}$ , for each eigenvalue  $-1, 0, 1$  of the triality operator  $\mathfrak{t}_\Lambda$ .*

**Proof.** Take the CAR-algebra  $\mathcal{C}$  given by

$$\{\mathfrak{a}^{*k}, \mathfrak{a}_k | k = 1, 2, \dots, n\}, \tag{3.37}$$

and fulfilling canonical anticommutation relations

$$[\mathfrak{a}^{*k}, \mathfrak{a}_l]_+ = \delta^k_l. \tag{3.38}$$

Denote its unique Hilbert representation space by  $H$  and define

$$j_l^k = a^{*k} a_l, \quad (3.39)$$

$$w_{pqr} = a_p a_q a_r, \quad (3.40)$$

$$w^{*ijk} = a^{*k} a^{*j} a^{*i}. \quad (3.41)$$

These abstract elements fulfil, of course, all relations of  $\mathcal{A}$  listed above. Thus,  $H$  carries a representation of  $\mathcal{A}$ . Since  $t_\Lambda$  commutes with all  $j$ 's,  $w$ 's and  $w^*$ 's,  $H$  decomposes into superselection sectors,

$$H = H_{-1} \oplus H_0 \oplus H_1,$$

corresponding to different eigenvalues of  $t_\Lambda$ . Each of these subspaces is invariant under the action of  $j$ 's,  $w$ 's and  $w^*$ 's, providing a nontrivial, faithful and irreducible representation of  $\mathcal{A}$ .  $\square$

**Theorem 3.3.** *Any irreducible, nontrivial representation of  $\mathcal{A}$  is equivalent to one of the three irreducible representations provided by lemma 3.2.*

For purposes of the proof, let us denote

$$E_{v_1, v_2, \dots, v_n} := (j^1_1)^{v_1} (i^1_1)^{v_1+1} (j^2_2)^{v_2} (i^2_2)^{v_2+1} \dots (j^n_n)^{v_n} (i^n_n)^{v_n+1}, \quad (3.42)$$

where all indices  $v_k$  assume values 0 or 1 and the summation is meant *modulo 2*. Since the  $i$ 's and  $j$ 's are Hermitian, orthogonal and commuting projectors  $\{E_{v_1, v_2, \dots, v_n}\}$  is a family of Hermitian orthogonal and commuting projectors too. Moreover, we have an obvious

**Corollary 3.4.** *The above projectors sum up to the unit element:*

$$\bigoplus E_{v_1, v_2, \dots, v_n} = \mathbf{1}. \quad (3.43)$$

**Lemma 3.5.** *The following relations hold for arbitrary  $k \neq l$ :*

1.  $j_l^k \cdot E_{v_1, v_2, \dots, v_n} = 0$  unless  $v_k = 0$  and  $v_l = 1$ .
2.  $E_{v_1, v_2, \dots, v_n} \cdot j_l^k = 0$  unless  $v_k = 1$  and  $v_l = 0$ .
3. For  $v_k = 0$  and  $v_l = 1$  we have  $j_l^k \cdot E_{v_1, \dots, v_k, \dots, v_l, \dots, v_n} = E_{v_1, \dots, v_k+1, \dots, v_l-1, \dots, v_n} \cdot j_l^k$

**Proof.** The proof follows by direct inspection from the following identities (which are all simple consequences of (3.17) and (3.18)):

$$j_l^k j_l^l j_k^k = 0, \quad j_l^k j_l^l i_k^k = j_l^k, \quad j_l^k i_l^l j_k^k = 0, \quad j_l^k i_l^l i_k^k = 0$$

and

$$j_l^l j_k^k j_l^k = 0, \quad j_l^l i_k^k j_l^k = 0, \quad i_l^l j_k^k j_l^k = j_l^k, \quad i_l^l i_k^k j_l^k = 0. \quad \square$$

**Proof of the theorem.** Take such an irreducible representation. Since the triality operator  $t_\Lambda$  lies in the centre of  $\mathcal{A}$ , it corresponds to a fixed value of triality. Take any other two irreducible representations, corresponding to the remaining values of triality. Let us denote these three representations by  $\mathcal{H}_t$ , with  $t = -1, 0, 1$ . We are going to prove that there exist isomorphisms

$$U_t : \mathcal{H}_t \rightarrow H_t \quad (3.44)$$

intertwining the representations  $\mathcal{H}_t$  with the three CAR representations  $H_t$ , defined in lemma 3.2. This will be accomplished by defining operators

$$c^{*k} : \mathcal{H}_t \rightarrow \mathcal{H}_{t+1}$$

and

$$c_k : \mathcal{H}_t \rightarrow \mathcal{H}_{t-1}$$

(with summation *modulo* 3), fulfilling the CAR and such that equations (3.39)–(3.41) are satisfied with  $a$ 's replaced by  $c$ 's. Then, the statement of the theorem is a consequence of the classical uniqueness theorem for CAR representations.

Let us denote

$$\mathcal{H}_{v_1, v_2, \dots, v_n} := E_{v_1, v_2, \dots, v_n} \mathcal{H}_t. \tag{3.45}$$

We obviously have

$$t_\Lambda E_{v_1, v_2, \dots, v_n} = E_{v_1, v_2, \dots, v_n} t_\Lambda = t E_{v_1, v_2, \dots, v_n}.$$

On the other hand, formula (3.13) implies

$$t_\Lambda E_{v_1, v_2, \dots, v_n} = \left( \sum_{i=1}^n v_i \bmod 3 \right) E_{v_1, v_2, \dots, v_n}.$$

Thus, the only non-trivial subspaces are those fulfilling the condition

$$t = \sum_{i=1}^n v_i \bmod 3. \tag{3.46}$$

This fact, together with (3.43), implies

$$\mathcal{H}_t = \bigoplus_{t = \sum_{i=1}^n v_i \bmod 3} \mathcal{H}_{v_1, v_2, \dots, v_n}. \tag{3.47}$$

Now, lemma 3.5 implies that  $j^k_l \mathcal{H}_{v_1, v_2, \dots, v_n} = 0$ , unless  $v_k = 0$  and  $v_l = 1$  and that, in the latter case,  $j^k_l$  maps  $\mathcal{H}_{v_1, \dots, v_k, \dots, v_l, \dots, v_n}$  onto  $\mathcal{H}_{v_1, \dots, v_k+1, \dots, v_l-1, \dots, v_n}$ . Observe, that  $j^k_l$  is an isomorphism of these two Hilbert spaces, with the inverse given by  $j^l_k$ . Indeed, we have:  $j^l_k j^k_l - j^k_l j^l_k = j^l_l - j^k_k$  and, consequently,

$$(j^k_l)^* j^k_l = j^l_k j^k_l = j^l_l - j^k_k + j^k_l j^l_k.$$

By lemma 3.5, this gives, for  $v_k = 0$  and  $v_l = 1$ ,

$$(j^k_l)^* j^k_l E_{v_1, \dots, v_k, \dots, v_l, \dots, v_n} = E_{v_1, \dots, v_k, \dots, v_l, \dots, v_n}. \tag{3.48}$$

Similarly, relations (3.23) imply that  $w_{lmn} \mathcal{H}_{v_1, \dots, v_l, \dots, v_m, \dots, v_n, \dots} = 0$ , unless  $v_l = v_m = v_n = 1$  and, in the latter case, it maps  $\mathcal{H}_{v_1, \dots, v_l, \dots, v_m, \dots, v_n, \dots}$  onto  $\mathcal{H}_{v_1, \dots, v_l-1, \dots, v_m-1, \dots, v_n-1, \dots}$ . Observe that, according to (3.9) and (3.10),  $w_{lmn}$  is an isomorphism of these two Hilbert spaces, with the inverse given by  $w^{*lmn}$ .

Since the representations  $\mathcal{H}_t$  are non-trivial, there is at least one non-vanishing vector in at least one of the subspaces  $\mathcal{H}_{v_1, v_2, \dots, v_n}$ , for every  $\mathcal{H}_t$ . Thus, let us choose three such *normalized* vectors and denote them by

$$|v_1(t), v_2(t) \dots v_n(t)\rangle \in \mathcal{H}_{v_1(t), v_2(t) \dots v_n(t)},$$

where  $\sum_{i=1}^n v_i(t) \bmod 3 = t$ ,  $t = -1, 0, 1$ . Acting with operators  $j^k_l$ ,  $w_{lmn}$  and  $w^{*lmn}$  on each of these three vectors, we obtain, for every  $t$ , a normalized vector, say  $|v_1, v_2 \dots v_n\rangle$ , in each of the subspaces  $\mathcal{H}_{v_1, \dots, v_l, \dots, v_m, \dots, v_n, \dots}$ . (The information about  $t$  is encoded implicitly, see equation (3.46).) Moreover, we can label the vectors in the representation spaces  $\mathcal{H}_{v_1, v_2, \dots, v_n}$  in

such a way that the following relations are fulfilled:

$$j^k_l | \dots, v_k, \dots, v_l, \dots \rangle = \begin{cases} \sigma(k, l) \cdot | \dots, v_k + 1, \dots, v_l - 1, \dots \rangle \\ \text{if } v_k = 0, v_l = 1, \\ 0 \text{ otherwise,} \end{cases}$$

$$\mathfrak{w}_{lmn} | \dots, v_l, \dots, v_m, \dots, v_n, \dots \rangle = \begin{cases} \sigma(l, m, n) \cdot | \dots, v_l - 1, \dots, v_m - 1, \dots, v_n - 1, \dots \rangle \\ \text{if } v_l = v_m = v_n = 1, \\ 0 \text{ otherwise,} \end{cases}$$

$$\mathfrak{w}^{*lmn} | \dots, v_l, \dots, v_m, \dots, v_n, \dots \rangle = \begin{cases} -\sigma(l, m, n) \cdot | \dots, v_l + 1, \dots, v_m + 1, \dots, v_n + 1, \dots \rangle \\ \text{if } v_l = v_m = v_n = 0, \\ 0 \text{ otherwise,} \end{cases}$$

where

$$\sigma(k, l) = (-1)^{s(k)-s(l)},$$

$$s(k) = \sum_{i < k} v_i,$$

$$\sigma(l, m, n) = s(l, m, n) (-1)^{s(l)+s(m)+s(n)},$$

and  $s(l, m, n)$  is the sign of the permutation which is necessary to sort the triple  $(l, m, n)$  in growing order (i.e.  $s(l, m, n) = 1$  if  $l < m < n$ ,  $s(l, m, n) = -1$  if  $l < n < m$ , etc).

We show that this labelling is possible, indeed: we start with three arbitrarily chosen vectors given by sequences of  $v_i = v_i(t)$ , for  $t = -1, 0, 1$ . Next, we apply operators  $j^k_l$ ,  $\mathfrak{w}_{lmn}$  and  $\mathfrak{w}^{*lmn}$  to these vectors and use the above formulae as the definition of the corresponding vectors on the right-hand side. Now, it remains to prove that this definition does not depend upon the order of these operations. For this purpose, we use the commutation rules (3.5) and (3.6). As far as the commutation relations  $[\mathfrak{w}, \mathfrak{w}]$ ,  $[\mathfrak{w}^*, \mathfrak{w}]$  and  $[\mathfrak{w}^*, \mathfrak{w}^*]$  are concerned, we can use relations (3.20)–(3.22) to flip the indices of occurring  $\mathfrak{w}$ 's and  $\mathfrak{w}^*$ 's in such a way that, whenever these objects meet, they have always the same indices. Then, we use relations (3.9) and (3.10) together with nilpotency properties (3.27) and (3.28). Having done this, the formula may be checked by inspection.

Because of the irreducibility of the representations  $\mathcal{H}_t$ , the vectors  $|v_1(t), v_2(t) \dots v_n(t)\rangle$  form (orthogonal) bases in each  $\mathcal{H}_t$ . Hence, we define the intertwining operator  $U$  putting

$$U |v_1, v_2 \dots v_n\rangle := (\mathfrak{a}^{*1})^{v_1} (\mathfrak{a}^{*2})^{v_2} \dots (\mathfrak{a}^{*n})^{v_n} |0\rangle,$$

where  $\mathfrak{a}^*$ 's are the CAR-creation operators from lemma 3.2 and  $|0\rangle \in H$  is the Fock vacuum. (The label  $t$  has been omitted.) Then, the operators

$$\mathfrak{c}^* := U^{-1} \mathfrak{a}^* U$$

and

$$\mathfrak{c} := U^{-1} \mathfrak{a} U$$

satisfy the CAR. It is easy to check that they fulfil equations (3.39)–(3.41), with  $\mathfrak{a}$  replaced by  $\mathfrak{c}$ . This ends the proof.  $\square$

This theorem shows that any algebra  $\mathcal{A}$  generated by abstract elements  $j, \mathfrak{w}$  and  $\mathfrak{w}^*$ , fulfilling relations (3.1)–(3.10), is isomorphic to the commutant of the triality operator

$$\mathfrak{t} = \varphi \left( \sum_k \mathfrak{a}^{*k} \mathfrak{a}_k \right)$$

in  $\mathcal{C}$ ,

$$\mathcal{A} \cong \mathfrak{t}'(\mathcal{C}) \subset \mathcal{C}. \tag{3.49}$$

This implies the following:

**Corollary 3.6.** *The algebras  $\mathcal{A}$  and  $\mathfrak{D}_T^{\text{mat}}$  are isomorphic.*

### 3.4. Super Lie structure

Formula (3.49) provides us with a simple and nice algebraic characterization of  $\mathfrak{D}_T^{\text{mat}}$ . Nonetheless, since in the case of lattice QED, we have found a Lie algebraic characterization of the matter field part [15, 16], it is worthwhile to ask whether a similar characterization is possible in QCD as well. The answer is affirmative, as we show now.

Using an idea of Palev [24], see also Dondi and Jarvis [25], we define the following operators:

$$\mathfrak{b}^{*k} := \phi^{*k} \sqrt{p - \mathfrak{n}}, \tag{3.50}$$

$$\mathfrak{b}_k := \sqrt{p - \mathfrak{n}} \phi_k, \tag{3.51}$$

with  $p$  being a positive integer. In what follows we use the following obvious formulae:

$$\phi_k f(\mathfrak{n}) = f(\mathfrak{n} + 1) \phi_k, \tag{3.52}$$

$$\phi^{*k} f(\mathfrak{n}) = f(\mathfrak{n} - 1) \phi^{*k}, \tag{3.53}$$

for any operator function  $f$ . In terms of the  $\mathfrak{b}$ -operators, the (anti-)commutation relations take the following form:

$$[j_l^k, j_n^m] = \delta_l^m j_n^k - \delta_n^k j_l^m, \tag{3.54}$$

$$[j_l^k, \mathfrak{b}^{*i}] = \delta_l^i \mathfrak{b}^{*k}, \tag{3.55}$$

$$[j_l^k, \mathfrak{b}_i] = -\delta_i^k \mathfrak{b}_l, \tag{3.56}$$

$$[\mathfrak{b}^{*k}, \mathfrak{b}_l]_+ = (p - \mathfrak{n}) \delta_l^k + j_l^k. \tag{3.57}$$

This shows that

$$\mathfrak{A} := \text{lin. env.} \{ \mathfrak{b}^{*k}, \mathfrak{b}_k, j_l^k \mid k, l = 1, 2, \dots, n \} \tag{3.58}$$

is isomorphic to the Lie superalgebra  $sl(1/n)$ . In more detail, identifying

$$e^k_l = j_l^k - \frac{1}{N} \delta_l^k \mathfrak{n}, \quad e^0_0 = \frac{N}{N-1} p - \mathfrak{n}, \quad e^k_0 = \mathfrak{b}^{*k}, \quad e^0_k = \mathfrak{b}_k, \tag{3.59}$$

we obtain the standard (anti-)commutation relations for  $sl(1/n)$ :

$$[e^k_l, e^m_n] = \delta_l^m e^k_n - \delta_n^k e^m_l,$$

$$[e^0_0, e^i_0] = -e^i_0,$$

$$[e^0_0, e^0_i] = e^0_i,$$

$$[e^k_l, e^i_0] = \delta_l^i e^k_0 - \frac{1}{N} \delta_l^k e^i_0,$$

$$[e^k_l, e^0_i] = -\delta_i^k e^0_l + \frac{1}{N} \delta_l^k e^0_i,$$

$$[e^k_0, e^0_l]_+ = e^k_l + \frac{N-1}{N} \delta_l^k e^0_0.$$

The even part is isomorphic to  $gl(n, \mathbb{C})$ ,

$$sl(1/n)_{\bar{0}} = gl(n, \mathbb{C}) = \text{lin. env. } \{e^k_l, e^0_0 \mid k, l = 1, 2, \dots, n\},$$

and the odd part is given by

$$sl(1/n)_{\bar{1}} = \text{lin. env. } \{e^k_0, e^0_k \mid k = 1, 2, \dots, n\}.$$

Next, observe that

$$\mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_k = \sqrt{F(n)} \mathfrak{w}_{ijk}, \quad (3.60)$$

with

$$F(n) = (p - n)(p - 1 - n)(p - 2 - n).$$

From now on we assume

$$p = n + 3.$$

Then  $F(n)$  is a positive operator in every representation. Thus, in every representation we can express the baryonic invariants  $\mathfrak{w}$  in terms of the fermionic operators  $\mathfrak{b}$ :

$$\mathfrak{w}_{ijk} = F(n)^{-\frac{1}{2}} \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_k. \quad (3.61)$$

We denote

$$\tilde{\mathfrak{w}}_{ijk} = \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_k, \quad (3.62)$$

and

$$\tilde{\mathfrak{w}}^{*ijk} = \mathfrak{b}^{*k} \mathfrak{b}^{*j} \mathfrak{b}^{*i}. \quad (3.63)$$

For the bosonic part, we implement relations (3.1) and (3.8)

$$(j^k_l)^* = j^l_k, \quad (3.64)$$

$$(j^k_k)^2 = j^k_k. \quad (3.65)$$

see subsection 3.1. These relations define a Lie ideal  $\mathfrak{J}$  in the enveloping algebra

$$\mathfrak{U}(gl(n, \mathbb{C})) \subset \mathfrak{U}(sl(1/n)),$$

by which we factorize. Moreover, we require that every observable has to commute with the triality operator. Thus, we have to take the commutant of  $\mathfrak{t}$  in this factor algebra, which we denote by

$$\mathcal{L} := \mathfrak{t}'(\mathfrak{U}(sl(1/n))/\mathfrak{J}). \quad (3.66)$$

**Theorem 3.7.** *The associative unital  $*$ -algebras  $\mathcal{A}$  and  $\mathcal{L}$  are isomorphic.*

**Proof.** First, observe that the operations of taking the commutant and of factorizing with respect to  $\mathfrak{J}$  commute, because  $\mathfrak{t}$  commutes with every  $j^k_k$ .

To prove the above isomorphism, we show that  $\mathcal{A}$  and  $\mathcal{L}$  have exactly the same irreducible representations. For that purpose, recall that  $sl(1/n)$  is a basic Lie superalgebra of type I, which means

$$sl(1/n)_{\bar{1}} = sl(1/n)_{-1} \oplus sl(1/n)_{+1}, \quad (3.67)$$

with  $sl(1/n)_{-1}$  and  $sl(1/n)_{+1}$  being two irreducible modules of  $sl(1/n)_{\bar{0}} \cong gl(n, \mathbb{C})$ , in terms of our generators spanned by  $\{\mathfrak{b}_k\}$  and  $\{\mathfrak{b}^{*k}\}$ , respectively. It follows from general representation theory, see [26], that any finite-dimensional irreducible representation of a

basic Lie superalgebra  $\mathfrak{G}$  is obtained from a Kac module. For superalgebras of type I, every Kac module  $V(\lambda)$  is induced from a highest weight module  $V_0(\lambda)$  of the even part  $\mathfrak{G}_0$ :

$$V(\lambda) = \text{Ind}_{\mathfrak{K}}^{\mathfrak{G}} V_0(\lambda) := \mathfrak{U}(\mathfrak{G}) \otimes_{\mathfrak{U}(\mathfrak{K})} V_0(\lambda), \tag{3.68}$$

where

$$\mathfrak{K} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \tag{3.69}$$

and  $\mathfrak{U}(\mathfrak{G})$  and  $\mathfrak{U}(\mathfrak{K})$  denote the enveloping algebras of  $\mathfrak{G}$  and  $\mathfrak{K}$ , respectively. Formula (3.68) has to be understood as follows: the  $\mathfrak{G}_0$ -module  $V_0(\lambda)$  has been extended to a  $\mathfrak{K}$ -module by putting

$$\mathfrak{G}_1 V_0(\lambda) = 0$$

and one has to identify elements

$$k \otimes v = 1 \otimes k(v),$$

for  $k \in \mathfrak{K}$  and  $v \in V_0(\lambda)$ . Then the induced representation of  $\mathfrak{G}$  is defined by

$$g(u \otimes v) := gu \otimes v, \tag{3.70}$$

for  $g \in \mathfrak{G}$ ,  $u \in \mathfrak{U}(\mathfrak{G})$  and  $v \in V_0(\lambda)$ . We stress that  $V(\lambda)$  is not always simple. In that case, one has to factorize by a certain maximal submodule, to obtain an irreducible representation.

Now, let  $V_0(\lambda)$  be a highest weight module of  $\mathfrak{G}_0 = gl(n, \mathbb{C})$ . Since in our case  $[\mathfrak{G}_{-1}, \mathfrak{G}_{-1}]_+ = 0$ , we have

$$V(\lambda) \cong \Lambda(\mathfrak{G}_{-1}) \otimes V_0(\lambda), \tag{3.71}$$

with

$$\Lambda(\mathfrak{G}_{-1}) = \bigoplus_{k=0}^n \Lambda^k(\mathfrak{G}_{-1})$$

denoting the exterior algebra of  $\mathfrak{G}_1$ . Thus, in terms of generators, we have

$$V(\lambda) \cong \bigoplus_{1 < k_1 < \dots < k_n \leq n} \mathfrak{b}_{k_1} \cdots \mathfrak{b}_{k_n} V_0(\lambda). \tag{3.72}$$

We show that taking the above commutant and factorizing with respect to  $\mathfrak{J}$  reduces the set of irreducible representations to three inequivalent representations labelled by triality.

First, in the commutant of  $\mathfrak{t}$ , only monomials in  $\mathfrak{b}$  and  $\mathfrak{b}^*$  built from  $\tilde{\mathfrak{w}}$  and  $\tilde{\mathfrak{w}}^*$  can occur. Thus,  $V(\lambda)$  takes the form

$$V(\lambda) \cong \bigoplus_{1 < i_1 < j_1 < k_1 < \dots < i_n < j_n < k_n \leq N} \tilde{\mathfrak{w}}_{i_1 j_1 k_1} \cdots \tilde{\mathfrak{w}}_{i_n j_n k_n} V_0(\lambda). \tag{3.73}$$

Since the  $\tilde{\mathfrak{w}}$ 's act transitively on this direct sum,  $V(\lambda)$  is an irreducible module. Moreover, as a direct consequence of the commutation relations we have

$$[\mathfrak{n}, j^k_l] = 0, \tag{3.74}$$

$$[\mathfrak{n}, \tilde{\mathfrak{w}}^{*ijk}] = 3 \tilde{\mathfrak{w}}^{*ijk}, \tag{3.75}$$

$$[\mathfrak{n}, \tilde{\mathfrak{w}}_{pqr}] = -3 \tilde{\mathfrak{w}}_{pqr}. \tag{3.76}$$

Thus, in any representation,  $\tilde{\mathfrak{w}}_{pqr}$  lowers the particle number by 3, whereas  $\tilde{\mathfrak{w}}^{*ijk}$  raises it by 3.

Next, by (3.65) the particle number operator  $j^k_k$  at position  $k$  can take only eigenvalues 0 and 1, on any highest weight module  $V_0(\lambda)$  of  $gl(n, \mathbb{C})$ . Every highest weight module of



$gl(n, \mathbb{C})$  is built—by taking tensor products—from fundamental representations, which in turn are all isomorphic to some exterior product  $\Lambda^l(\mathbb{C}^n)$ . But, whenever we take a tensor product of such exterior products, which is not antisymmetric, there exists a vector, for which  $j^k_k$  has an eigenvalues greater than 1. Thus, (3.65) reduces the admissible highest weight modules to the fundamental ones. Since the operators  $\tilde{\mathfrak{m}}$  lower the particle number by 3, the lowest weight component of (3.73) can have particle numbers 0, 1 or 2 only. Using the canonical basis of  $\mathbb{C}^n$ , an explicit isomorphism intertwining these three representations with the representations  $H_t$  can be written down, as in the proof of theorem 3.3.  $\square$

**4. Discussion**

1. The generators  $J_\gamma^{ab}(x, y)$  and  $W_{\alpha\beta\gamma}^{abc}(x, y, z)$  (see (2.10) and (2.11) are difficult to handle. This is why we have replaced them by generators  $j^k_l$  and  $\mathfrak{m}_{pqr}$  (see (2.17) and (2.19), fulfilling much simpler relations. To define these observables we have used a gauge fixing procedure based upon the choice of a tree. However, it is obvious that the specific gauge we have chosen is irrelevant for the structure of the algebra, defined by relations (3.1)–(3.10). Changing the gauge condition does not affect these relations. Thus, there should exist another, more intrinsic procedure for obtaining this algebra, which does not rely on gauge fixing.

To make this transparent, assume that we have chosen a tree. Now, instead of fixing the gauge, we rewrite the operators  $J$  and  $W$  in terms of fermionic operators parallel transported to the tree root  $x_0$ . Denoting these transported operators by  $\tilde{\psi}$ , we get

$$J_\gamma^{ab}(x, y) = \tilde{\psi}_A^{*a}(x_0)U_{\sigma B}^A \tilde{\psi}^{bB}(x_0), \tag{4.1}$$

with

$$\sigma = \beta \circ \gamma \circ \alpha^{-1}$$

being the closed path uniquely defined by this parallel transport ( $\alpha$  and  $\beta$  are the unique on-tree paths from  $x$  resp.  $y$  to  $x_0$ .) Thus, the operators  $J$  acquire a labelling by (unparametrized) closed paths. Collecting the spinorial index and the point  $x \in \Lambda$  into a single index  $u = (a, x)$ , we get a mapping  $\sigma \mapsto J^u_v(\sigma)$ . It can be easily checked that the commutation relations for the quantities  $J$  then take the following form:

$$[J^u_v(\beta), J^w_t(\gamma)] = \delta^w_v J^u_t(\beta \circ \gamma) - \delta^u_t J^w_v(\gamma \circ \beta), \tag{4.2}$$

where  $\circ$  denotes the natural multiplication in the group of (unparametrized) closed paths. Similarly, the baryonic operators  $W$  and  $W^*$  can be rewritten, acquiring a labelling by closed paths,  $\sigma \mapsto W^{uvw}(\sigma)$ ,  $\sigma \mapsto W^{*uvw}(\sigma)$ . The anti-commutation relations for  $W$  and  $W^*$  can be easily worked out, but we omit them here.

2. Now, let us restrict ourselves to on-tree paths in  $J$  and  $W$  only. It is clear from (4.1) that for them all on-tree parallel transporters  $U_\sigma$  are equal to  $\mathbf{1}$ , yielding quantities  $J^u_v, W_{uvw}, W^{*uvw}$ . Thus, the algebra labelled by closed paths descends to an algebra defined by the following (anti-)commutation relations:

$$[J^u_v, J^w_t] = \delta^w_v J^u_t - \delta^u_t J^w_v, \tag{4.3}$$

$$\{W^{*(uvw)}, W_{(pqr)}\} = \frac{1}{2}(J \cdot J \cdot \delta)^{(uvw)}_{(pqr)} + 2(J \cdot \delta \cdot \delta)^{(uvw)}_{(pqr)} - 6\delta^{(uvw)}_{(pqr)}, \tag{4.4}$$

$$\{W^{*(uvw)}, W^{*(pqr)}\} = 0, \quad \{W_{(uvw)}, W_{(pqr)}\} = 0, \tag{4.5}$$

$$[J^s_t, W^{*(uvw)}] = \delta_t^u W^{*(svw)} + \delta_t^v W^{*(usw)} + \delta_t^w W^{*(uvs)}, \tag{4.6}$$

$$[J^s_t, W_{(pqr)}] = -\delta^s_p W_{(tqr)} - \delta^s_q W_{(ptr)} - \delta^s_r W_{(pqt)}, \tag{4.7}$$

Here,  $(J \cdot J \cdot \delta)$  and  $(J \cdot \delta \cdot \delta)$  are the appropriate totally symmetric combinations. Allowing for cyclic permutations on  $uvw$  and  $pqr$ ,  $(J \cdot J \cdot \delta)$  contains a total of  $4 \times 9 = 36$  terms,  $(J \cdot \delta \cdot \delta)$  contains  $9 \times 2 = 18$  terms and  $(\delta \cdot \delta \cdot \delta)$  just  $3 \times 2 = 6$  terms:

$$\begin{aligned} (J \cdot J \cdot \delta)_{(pqr)}^{(uvw)} &= (J^u_p J^v_q + J^v_p J^u_q + J^u_q J^v_p + J^v_q J^u_p) \delta^w_r + \dots, \\ (J \cdot \delta \cdot \delta)_{(pqr)}^{(uvw)} &= J^u_p (\delta^v_q \delta^w_r + \delta^v_r \delta^w_q) + \dots, \\ \delta^{(uvw)}_{(pqr)} &= \delta^u_p (\delta^v_q \delta^w_r + \delta^v_r \delta^w_q) + \dots. \end{aligned} \tag{4.8}$$

Obviously, equations (4.3) are the commutation relations of  $gl(4N, \mathbb{C})$ . The anti-commutator (4.4) closes on a quadratic polynomial in the enveloping algebra of the even (Lie) subalgebra  $gl(4N, \mathbb{C})$ . Thus, we have identified the  $W$  and  $W^*$  as odd generators of a supersymmetry algebra belonging to a class of ‘polynomial’ superalgebras. Such ‘nonlinear’ extensions of Lie algebras and superalgebras have been recognized in other contexts in recent literature. An initial investigation of them in the case of generalizations of  $gl(4N/1)$  (or more generally of type I Lie superalgebras) has been given in [13] (see also the related remarks in the appendix).

We stress that, again by formula (4.1), the full set of operators  $J$  and  $W$  can be reconstructed, knowing the generators  $J^u_v, W^{uvw}, W^{*uvw}$ , together with the Wilson loops  $U_\sigma$ .

- Clearly, the quantities  $J^u_v, W_{uvw}, W^{*uvw}$  constructed under point (2) can be viewed as obtained from on-tree gauge fixing (putting the parallel transporter on every on-tree link equal to **1**). If we remove the residual gauge freedom (at the root), we can pass to the quantities  $j^k_l, w_{ijk}, w^{*pqr}$  used in this paper. In the case of a generic orbit we have proved (see [6]) that the representation  $J^u_v(\beta)$  is ‘sufficiently non-degenerate’, and we may reduce it to  $j^k_l$ . Actually, this non-degeneracy follows from the non-degeneracy of the representation of the electric fluxes  $E_\gamma(x, y)$ , see (2.9). We expect that there exist ‘degenerate’ representations, related to non-generic orbits, which do not allow to extract the representation of the full  $gl(n, \mathbb{C})$  Lie algebra. Indeed, the impossibility to fix the gauge completely on a non-generic orbit (having a non-trivial stabilizer) implies the impossibility of reconstructing the quantities  $j^k_l$ , because they are not invariant with respect to the stabilizer. In this case, we expect that the fermions  $\alpha_k$  carrying the representation (see formulae (3.37)–(3.40)) will be replaced by some ‘anyons’, satisfying (possibly) a different statistics. A consistent mathematical analysis of such representations of the observable algebra (if they do exist) together with their physical implications will be one of our next goals.

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### Appendix A. Appendix. Additional relations

Here, we list additional relations, also following from relations (3.1)–(3.10).

First, we have the following so-called characteristic identities:

$$j^k_l j^l_m = (n+1-n)j^k_m, \quad (\text{A.1})$$

(with the sum taken over all  $l$ ). Next, one can analyse arbitrary higher order monomials, built from  $\mathfrak{w}$  and  $\mathfrak{w}^*$ . For that purpose, let us introduce the following tensor operators (totally antisymmetric in both upper and lower indices) built from  $j$ 's:

$$X^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} \text{sgn}(\rho) \text{sgn}(\sigma) j^{i_{\rho_1}}_{p_{\sigma_1}} j^{i_{\rho_2}}_{p_{\sigma_2}} j^{i_{\rho_3}}_{p_{\sigma_3}}, \quad (\text{A.2})$$

$$Y^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} \text{sgn}(\rho) \text{sgn}(\sigma) j^{i_{\rho_1}}_{p_{\sigma_1}} j^{i_{\rho_2}}_{p_{\sigma_2}} \delta^{i_{\rho_3}}_{p_{\sigma_3}}, \quad (\text{A.3})$$

$$Z^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} \text{sgn}(\rho) \text{sgn}(\sigma) j^{i_{\rho_1}}_{p_{\sigma_1}} \delta^{i_{\rho_2}}_{p_{\sigma_2}} \delta^{i_{\rho_3}}_{p_{\sigma_3}}. \quad (\text{A.4})$$

$$D^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} \text{sgn}(\rho) \text{sgn}(\sigma) \delta^{i_{\rho_1}}_{p_{\sigma_1}} \delta^{i_{\rho_2}}_{p_{\sigma_2}} \delta^{i_{\rho_3}}_{p_{\sigma_3}}, \quad (\text{A.5})$$

with sums running over all permutations  $\rho$  and  $\sigma$ . Using (2.19) and (2.20), a lengthy but simple calculation yields

$$36\mathfrak{w}^{*ijk} \mathfrak{w}_{pqr} = X^{ijk}_{pqr} + 3Y^{ijk}_{pqr} + 2Z^{ijk}_{pqr}, \quad (\text{A.6})$$

$$36\mathfrak{w}_{pqr} \mathfrak{w}^{*ijk} = -X^{ijk}_{pqr} + 6Y^{ijk}_{pqr} - 11Z^{ijk}_{pqr} + 6D^{ijk}_{pqr}. \quad (\text{A.7})$$

Using these relations and keeping in mind the nilpotency properties, one can calculate arbitrary (even order) polynomials in  $\mathfrak{w}$  and  $\mathfrak{w}^*$  in terms of the above tensor operators. In particular, taking the sum of these two relations, we get the following anticommutator for the baryonic observables:

$$[\mathfrak{w}^{*ijk}, \mathfrak{w}_{pqr}]_+ = \frac{1}{4}(Y^{ijk}_{pqr} - Z^{ijk}_{pqr} + \frac{2}{3}D^{ijk}_{pqr}). \quad (\text{A.8})$$

In fact, these relations (A.8) together with (3.5), (3.6), (3.7) and the mutual anticommutativity of  $\mathfrak{w}$  and  $\mathfrak{w}^*$  can again be taken as the defining relations for a type of polynomial superalgebra generalizing  $gl(12N/1)$ , this time with odd generators of antisymmetric type (see [13] for details, and also the discussion above).

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